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GEODESIC SPHERES AND TWO-POINT HOMOGENEOUS SPACES

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ABSTRACT

In the Osserman conjecture and in the isoparametric conjecture it is stated that two-point homogeneous spaces may be characterized via the constancy of the eigenvalues of the Jacobi operator or the shape operator of geodesic spheres, respectively. These conjectures remain open, but in this paper we give complete positive results for similar statements about other symmetric endomorphism fields on small geodesic spheres. In addition, we derive more characteristic properties for this class of spaces by using other properties of small geodesic spheres. In particular, we study Riemannian manifolds with (curvature) homogeneous geodesic spheres.

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1. Introduction

Let M be a connected, smooth Riemannian manifold and ∇ its Levi Civita connection. R denotes the associated Riemannian curvature tensor (with the convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$). For a unit tangent vector v the associated **Jacobi operator** R_v is the self-adjoint endomorphism $R_v := R(.,v)v$. These operators play a fundamental role in Riemannian geometry because, as can be seen by replacing v by the unit tangent vector field $\dot{\gamma}$ of a geodesic γ in M, they determine the Jacobi vector fields along geodesics which are important in curvature theory. For some classes of Riemannian manifolds these operators have particularly nice properties. For example, let M be a two-point homogeneous space. Since the isometry group acts transitively on the unit tangent sphere bundle, R_v has eigenvalues which are independent of v, that is, the eigenvalues are globally constant. This leads to the following

OSSERMAN CONJECTURE ([Os]): *A Riemannian manifold with globally constant eigenvalues for the Jacobi operators is locally isometric to a two-point homogeneous space.*

A space with globally constant eigenvalues for the Jacobi operators is called a **globally Osserman space** and it is said to be a **pointwise Osserman space** if the eigenvalues of R_v are independent of $v \in T_mM$ but may vary with $m \in M$. Chi proved in [Ch1], [Ch2] and [Ch3] that this conjecture holds in a lot of special cases (see below). We refer also to [GSV] where it is shown, by using self-dual four-dimensional Einstein spaces, that not every pointwise Osserman space is a globally Osserman space. Despite the effort made, the Osserman conjecture remains open.

Next, we note that the Jacobi operators are immediately related to the **shape operators** S_m of the geodesic spheres $G_m(r)$ with radius r and centered at m (see for example [Be], [CV] and [V2]). The consideration of the eigenvalues of these operators led to a well-known definition of a very interesting class of manifolds, namely the harmonic spaces: M is said to be **harmonic** if for each $m \in M$ all sufficiently small geodesic spheres are hypersurfaces with constant mean curvature, that is, $\operatorname{tr} S_m =: h_m$ is a radial function for each $m \in M$ (see [Be], [RWW], [V1] and [V2]). Two-point homogeneous spaces provide nice examples of harmonic spaces but, since the surprising discovery of the **DR-spaces** [DR] (see also [Sz3], [TV2]) we know that in contrast to the Lichnerowicz conjecture, they are not the only examples. Moreover, in [TV2] it is shown that for the non-symmetric DR-spaces, not all the eigenvalues of the shape operators (that is, the principal curvatures) are radial functions. This led to the

ISOPARAMETRIC CONJECTURE ([TV2]): *A Riemannian manifold is locally isometric to a two-point homogeneous space if and only if all sufficiently small geodesic spheres are isoparametric hypersurfaces (that is, the eigenvalues of the shape operators S_m are radial functions).*

Via the Gauss equation the shape operators S_m yield the **Ricci operators** $\tilde{Q}_m$ of the geodesic spheres G_m and it can be seen from several properties (see for example [CV], [DV] and [V2]) that $\tilde{Q}_m$ and S_m play a very parallel role in local differential geometry. This leads to the

INTRINSIC VERSION OF THE ISOPARAMETRIC CONJECTURE ([GSV]): *Let M, $\dim M > 2$, be a Riemannian manifold such that the eigenvalues of the Ricci operators of small geodesic spheres are radial functions. Then M is locally isometric to a two-point homogeneous space.*

It is proved in [GSV] that in the last two conjectures the spaces are necessarily globally Osserman spaces. Even this result and the fact that the spaces are also harmonic spaces did not enable us to give a full positive answer.

The first main purpose of the paper is to show that one may indeed characterize locally the two-point homogeneous spaces by the radial character of the eigenvalues of some other special symmetric endomorphisms related to geodesic spheres. In the second part of the paper we will provide several other characterizations by concentrating on other properties of geodesic spheres. More precisely, we shall use on the one hand a characterization by the Killing character of the Ricci tensor $\tilde{\rho}$ of type (0,2) or the second fundamental form $\tilde{\sigma}$ of geodesic spheres given in [CV] and on the other hand the theory of $\mathfrak{C}$- and $\mathfrak{P}$-spaces introduced in [BV]. In particular we discuss the manifolds with (curvature) homogeneous geodesic spheres.

2. Two-point homogeneous spaces and radial eigenvalues

Let M be an n-dimensional connected Riemannian manifold and let g, Q, ρ and τ denote the Riemannian metric, Ricci operator, Ricci tensor and scalar curvature of M, respectively. Further, for a unit tangent vector v of M, put

$R'_v := (\nabla_v R)(.,v)v$ and $R''_v := (\nabla^2_{vv} R)(.,v)v$. Next, let γ be a unit speed geodesic in M and put $R_\gamma := R(.,\dot\gamma)\dot\gamma$, $m := \gamma(0)$, $u := \dot\gamma(0)$ and $p := \exp_m(ru)$ for sufficiently small $r \in \mathbb{R}_+$. Let $S_m(p)$ and $h_m(p) = \operatorname{tr} S_m(p)$ denote the shape operator and mean curvature of the geodesic sphere $G_m(r)$ at p. Similarly, $\tilde{Q}_m(p)$ and $\tilde\tau(p) = \operatorname{tr} \tilde{Q}_m(p)$ denote the Ricci operator and scalar curvature of $G_m(r)$ at p. Finally, let I denote the identity transformation of T_mM.

In what follows we will always identify the tangent spaces T_pM and T_mM via parallel translation along γ. Sometimes we restrict the appearing operators to the orthogonal complement of $\mathbb{R}u$, but we shall use the same notation for the restricted operators as it becomes clear from the context when we actually restrict.

Lemma 1 ([CV]): *We have*

$$\begin{aligned}
(1)\ \tilde{Q}_m(p) - \frac{n}{r}S_m(p) = &-\frac{2}{r^2}I + \left[Q(m) - \rho(u,.)u - \frac{1}{3}\rho(u,u)I\right] \\
&+ r\left[\nabla_u Q - (\nabla_u\rho)(u,.)u - \frac{1}{4}(\nabla_u\rho)(u,u)I - \frac{1}{4}R'_u\right] \\
&+ O(r^2), \\
(2)\ \tilde{Q}_m(p) + \frac{n}{3}R_\gamma(p) = &\frac{n-2}{r^2}I + \left[Q(m) - \rho(u,.)u - \frac{1}{3}\rho(u,u)I\right] \\
&+ r\left[\nabla_u Q - (\nabla_u\rho)(u,.)u - \frac{1}{4}(\nabla_u\rho)(u,u)I + \frac{n-3}{12}R'_u\right] \\
&+ O(r^2), \\
(3)\ \ S_m(p) + \frac{r}{3}R_\gamma(p) = &\frac{1}{r}I + \frac{1}{12}r^2R'_u + \frac{1}{15}r^3\left[R''_u - \frac{1}{3}R_u^2\right] + O(r^4), \\
(4)\ \ \tilde\tau_m(p) - \frac{n}{r}h_m(p) = &-\frac{2(n-1)}{r^2} + \left[\tau(m) - \frac{n+2}{3}\rho(u,u)\right] + O(r).
\end{aligned}$$

In [Sz2] Szabó proved a nice property which played a key role in his proof of the fact that two-point homogeneous spaces are locally symmetric. This property will also be important in our proofs. For that reason we state it here explicitly.

Lemma 2 ([Sz2]): *Let M be a Riemannian manifold. Then M is locally symmetric if and only if at each point $m \in M$ the eigenvalues of R'_v are independent of v for all unit vectors $v \in T_mM$.*

See [Gi] for an extension of this result.

Now we state and prove the main result of this section.

THEOREM 1: *Let M be an n-dimensional connected Riemannian manifold. Then the following statements are equivalent:*

(i) *M is locally isometric to a two-point homogeneous space;*

(ii) *the operators $\tilde{Q}_m - \frac{n}{d}S_m$ have radial eigenvalues for all $m \in M$;*

(iii) *the operators $\tilde{Q}_m + \frac{n}{3}R_\gamma$ have radial eigenvalues for all $m \in M$;*

(iv) *the operators $S_m + \frac{d}{3}R_\gamma$ have radial eigenvalues for all $m \in M$.*

Here d denotes the distance to the center m.

Proof: First, we note that it is well-known and easy to prove that (i) implies (ii), (iii) and (iv). (See for example [V2].)

Next, we prove the converse and note first that $\tilde{Q} = 0$ for $n = 2$. In this case (iii) yields (i) at once since the sectional curvature is constant. Also (ii) yields (i) because for $n = 2$ (ii) means that M is harmonic and hence has constant scalar curvature [Be], [V2].

Now we prove that for $n \geq 3$ (ii) and (iii) imply that M is a harmonic manifold. We start with (ii). The hypothesis yields that

$$\operatorname{tr}\left(\tilde{Q}_m - \frac{n}{d}S_m\right) = \tilde{\tau}_m - \frac{n}{d}h_m \tag{5}$$

is a radial function. Then (4) implies easily that M is an Einstein space. (Note that in this case M is analytic in normal coordinates [DK].) Next, the contracted Gauss equation for $G_m(r)$ becomes, taking into account the Einstein property and hence the constancy of τ,

$$\tilde{\tau}_m = \frac{n-2}{n}\tau + h_m^2 - \operatorname{tr} S_m^2. \tag{6}$$

So, (5) and (6) yield that

$$h_m^2 - \frac{n}{d}h_m - \operatorname{tr} S_m^2 \tag{7}$$

is a radial function. Moreover, the well-known Riccati equation

$$S_m' + S_m^2 + R_\gamma = 0$$

for the shape operator S_m (see [Be], [CV], [Gr], [V2]) gives, by taking the trace and using (7), that

$$h_m' + h_m^2 - \frac{n}{d}h_m \tag{8}$$

is a radial function. Finally, because of the analyticity, we may use the power series expansion

$$h_m(p) = \frac{n-1}{r} + \sum_{k=1}^{\infty} \beta_k(m,u) r^k \tag{9}$$

for the mean curvature function. Then the condition on (8) and (9) imply that

$$-2(n-1) + \sum_{k=1}^{\infty}(n-2+k)\beta_k(m,u)r^{k+1} + \left(\sum_{k=1}^{\infty}\beta_k(m,u)r^{k+1}\right)^2$$

must be independent of u. So

$$(n-1+l)\beta_{l+1}(m,u) + \sum_{\substack{\lambda+\mu=l\\ \lambda,\mu\geq 1}} \beta_\lambda(m,u)\beta_\mu(m,u)$$

must be independent of u for $l \in \mathbb{N}$. Induction shows that all $\beta_k(m,u)$ are independent of u and hence h_m is a radial function. This means that M is harmonic.

To prove the same result for (iii) we note that in this case $\operatorname{tr}(\tilde{Q}_m + \frac{n}{3}R_\gamma)$ must be radial and so, from (2), $\tau(m) - \frac{n+2}{3}\rho(u,u)$ must be independent of u for all $m \in M$. Hence, M is again an Einstein space. Then (iii) yields that $\tilde{\tau}_m = \operatorname{tr}\tilde{Q}_m$ is radial and since $n > 2$, this means that the manifold is harmonic [CV].

Next, we prove that (ii), (iii) and (iv) imply that M is locally symmetric. We start with (ii). Since M is an Einstein space we get from (1)

$$\tilde{Q}_m - \frac{n}{d}S_m + \left(\frac{2}{d^2} - \frac{2\tau}{3n}\right) I = -\frac{1}{4}d\,R'_u + O(d^2)\,. \tag{10}$$

By taking the traces of the kth powers, $k = 1, \ldots, n-1$, of both members in (10) we see at once that $\operatorname{tr}(R'_u)^k$ is independent of u. So the result follows by using Lemma 2. A similar reasoning gives the same result for (iii) when $n \neq 3$. But for $n = 3$ M is also locally symmetric since a three-dimensional Einstein space has constant curvature. Finally, (iv) implies also the result but now it follows directly from (3).

We may conclude that this proves the equivalence of (ii), (iii) and (i) since a locally symmetric harmonic space is locally isometric to a two-point homogeneous space (see [Be], [V2], also for further references).

Now we finish the proof by showing that (iv) implies (i) and note that we know already that M is locally symmetric. So we may suppose $n \geq 3$. Then (3) becomes

$$S_m + \frac{d}{3}R_\gamma = \frac{1}{d}I - \frac{1}{45}d^3R_u^2 + O(d^4). \tag{11}$$

The hypothesis now implies that $\operatorname{tr} R_u^2$ must be independent of u. This is the second harmonicity condition (see [Be], [CV], [RWW], [V1], [V2]). In this case we have

(a) if M is (locally) reducible, it is flat [CV];

(b) if M is (locally) irreducible, it is an Einstein space (as M is locally symmetric) and then (iv) means that $\operatorname{tr} S_m = h_m$ is radial, that is, M is a harmonic space. This again yields the result and the proof is completed. ∎

3. Further characterizations

In this second part we always suppose that M is connected and of dimension $n \geq 2$. We shall give other local characterizations of two-point homogeneous spaces using special geometric properties of geodesic spheres. To do this let $\tilde{\rho}$, as before, denote the Ricci tensor and $\tilde{\sigma}$ the real-valued second fundamental form of a small geodesic sphere. Both tensors are symmetric and we have the useful

LEMMA 3 ([CV]): *A Riemannian manifold M is locally isometric to a two-point homogeneous space if and only if*

(i) *$\tilde{\sigma}$ is a Killing tensor for all small geodesic spheres in M, or*

(ii) *(for $n > 2$) $\tilde{\rho}$ is a Killing tensor for all small geodesic spheres in M.*

Note that a symmetric tensor field on a manifold is said to be a Killing tensor (or also cyclic-parallel) if the cyclic sum over all entries in the covariant derivative of the tensor vanishes. The geometrical meaning of (i) is that every geodesic in the sphere is a circle in the ambient space.

To derive some applications we first recall the definition of a $\mathfrak{C}$- and a $\mathfrak{P}$-space introduced in [BV]. A Riemannian manifold M is said to be a **$\mathfrak{C}$-space** if for every geodesic γ in M the eigenvalues of the Jacobi operator R_γ are constant along γ; and M is called a **$\mathfrak{P}$-space** if for every geodesic γ in M the Jacobi operator R_γ is diagonalizable by a parallel orthonormal frame field along γ. In [BV] it is proved that the intersection of the two classes of $\mathfrak{C}$- and $\mathfrak{P}$-spaces is precisely the class of locally symmetric spaces; moreover, several classes of $\mathfrak{C}$- and $\mathfrak{P}$-spaces consisting not only of locally symmetric spaces are provided there. In particular, any Riemannian homogeneous space M with the property that each geodesic is an orbit of a one-parameter subgroup of the isometry group of M (that is, a **Riemannian g.o. space** [KV2]) is a $\mathfrak{C}$-space. This class contains the naturally reductive Riemannian homogeneous spaces. Further, let G denote the connected component of the full isometry group of a Riemannian homogeneous space M and let $D(G/H)$ be the algebra of G-invariant differential operators on $M = G/H$. Then M is said to be a **commutative** or **Gelfand space** if $D(G/H)$ is a commutative algebra. In [BV] it is proved that all commutative spaces are $\mathfrak{C}$-spaces. We also refer to [BV] for examples of $\mathfrak{P}$-spaces and for further information.

Now we have

PROPOSITION 1: *A Riemannian manifold M^n, $n > 2$, is locally isometric to a two-point homogeneous space if and only if each small geodesic sphere is a $\mathfrak{C}$-space.*

Proof: First, let $G_m(r)$ be a $\mathfrak{C}$-space. Then its Ricci tensor is a Killing tensor [BV] and so the result follows from Lemma 3. Conversely, let M be a two-point homogeneous space. Then, an unpublished result of S. Helgason and the third author states that each geodesic sphere in M is a commutative space, and hence a $\mathfrak{C}$-space. ∎

Note that in fact the geodesic spheres in two-point homogeneous spaces are naturally reductive Riemannian homogeneous spaces except for the Cayley plane and its non-compact dual (see [V2] for further references).

As a related result we prove now

PROPOSITION 2: *A Riemannian manifold M^n, $n > 2$, is locally isometric to a two-point homogeneous space if and only if all small geodesic spheres in M are spaces with volume-preserving (local) geodesic symmetries (that is, D'Atri spaces [VW]).*

Proof: Let M be a two-point homogeneous space. Then the geodesic spheres have the required property since any commutative space is a space with volume-preserving (local) geodesic symmetries [KV1]. Conversely, suppose that all geodesic spheres in M are spaces with volume-preserving (local) geodesic symmetries.

Then the tensors $\tilde{\rho}$ are Killing tensors (see for example [V2]) and the assertion follows from Lemma 3. ∎

Next, we recall the notion of a curvature homogeneous Riemannian space. Following I.M. Singer [Si] a Riemannian manifold M is said to be **curvature homogeneous** if for each pair of points p and q in M there exists a linear isometry $F : T_pM \to T_qM$ such that $F^*R_q = R_p$. Of course, any (locally) homogeneous space is curvature homogeneous, but the converse is not true. See [KTV1], [KTV2] and [TV1] for more details and information. It is clear that small geodesic spheres in two-point homogeneous spaces are curvature homogeneous. As concerns the possible converse we have the following result which gives only a partial answer.

PROPOSITION 3: *Let M^n, $n > 2$, be a Riemannian manifold such that all its small geodesic spheres are curvature homogeneous. Then M is a harmonic globally Osserman space.*

Proof: The hypothesis implies that for each geodesic sphere in M the eigenvalues of its Ricci operator are radial functions. So $\tilde{\tau}$ is constant on geodesic spheres and hence M is a harmonic space [CV]. The assertion then follows from the note made after the statement of the intrinsic version of the isoparametric conjecture given in the introduction. ∎

COROLLARY 1: *Let M^n, $n > 2$, be a Riemannian manifold all of whose small geodesic spheres are homogeneous. Then M is a harmonic globally Osserman space.*

Note that we do not suppose here that the geodesic spheres are homogeneous with respect to isometries induced from those of the ambient space. If this would be the case then it would imply that M is locally isometric to a two-point homogeneous space. (A proof of this follows at once from (3) and Lemma 2.)

The difficulty in getting a complete result comes from the fact that the Osserman conjecture is still open even if we suppose the manifold is in addition a harmonic space. For up-to-date information about these two classes of spaces we include the next two propositions.

PROPOSITION 4 ([Sz1], [Sz3]): *Let M be a compact harmonic manifold satisfying one of the following conditions:*

(i) *M has a finite fundamental group;*

(ii) *M has non-negative scalar curvature.*

Then M is locally isometric to a two-point homogeneous space.

Note that (ii) follows by using the result for (i) and the Cheeger–Gromoll splitting theorem.

PROPOSITION 5 ([Ch1], [Ch2], [Ch3]): *Let M be an n-dimensional Riemannian manifold. Then the Osserman conjecture is true in the following cases:*

(i) $n \equiv 1 (\mathrm{mod}\, 2)$;

(ii) $n \equiv 2 (\mathrm{mod}\, 4)$;

(iii) $n = 4$;

(iv) $n = 4k$, $k = 2, 3, \ldots$ *and M is a simply connected compact quaternionic Kähler manifold with vanishing second Betti number;*

(v) *M satisfies the following axioms:*

   (a) *R_v has precisely two different constant eigenvalues independent of $v \in SM$ (the unit tangent sphere bundle of M);*

   (b) *let λ and μ be the two eigenvalues and for $v \in SM$ denote by $E_\mu(v)$ the span of v and the eigenspace of R_v with eigenvalue μ; then $E_\mu(w) = E_\mu(v)$ whenever $w \in E_\mu(v)$;*

(vi) *M is a Kähler manifold of non-negative or non-positive sectional curvature.*

Note that (v)(b) is redundant when $\dim E_\mu(v) = 2$. We refer to [Ch2] and [GSV] for more results in the quaternionic case.

COROLLARY 2: *Let M^n, $n > 2$, be a Riemannian manifold with curvature homogeneous geodesic spheres and satisfying one of the hypotheses given in Proposition 4 (+ M compact) or Proposition 5. Then M is locally isometric to a two-point homogeneous space.*

Further we have

COROLLARY 3: *Not all the small geodesic spheres in a non-symmetric DR-space M are curvature homogeneous. More precisely, for each $p \in M$ and each sufficiently small normal neighborhood U_p of p there exists an infinite number of geodesic spheres in U_p centered at p which are not curvature homogeneous.*

Proof: The result follows from Proposition 3 and the fact that the non-symmetric DR-spaces are homogeneous and not globally Osserman spaces [Sz3], [TV2]. ∎

Before giving the next characterizations we recall that a hypersurface N in M is said to be **curvature-adapted** if at each point of N the Jacobi operator of M and the shape operator of N with respect to a unit normal vector at that point commute. Then we have

PROPOSITION 6: *A Riemannian manifold M^n, $n > 2$, is locally isometric to a two-point homogeneous space if and only if one of the following conditions is satisfied:*

(i) *all small geodesic spheres in M are curvature-adapted and curvature homogeneous;*
(ii) *all small geodesic spheres in M are curvature-adapted isoparametric hypersurfaces;*
(iii) *M is a globally Osserman space with curvature-adapted geodesic spheres.*

Proof: First, let M be a two-point homogeneous space. Then it is clear that (i), (ii) and (iii) are satisfied. To prove the converse we first note that in the real analytic case M is a $\mathfrak{P}$-space if and only if all small geodesic spheres in M are curvature-adapted hypersurfaces [BV]. Moreover, it is clear that any globally Osserman space is a $\mathfrak{C}$-space. The results in [GSV] about the isoparametric conjecture and its intrinsic version imply that under the hypotheses in (ii) the space is a globally Osserman space. According to Proposition 3 M is also a globally Osserman space if (i) holds. So M is both a $\mathfrak{C}$- and a $\mathfrak{P}$-space and hence locally symmetric. This proves the assertion since any locally symmetric pointwise Osserman space is locally isometric to a two-point homogeneous space (see, for example, [GSV]). ∎

Note that the isoparametric conjecture just says that one could drop "curvature-adapted" in (ii), and the Osserman conjecture says that the phrase "with curvature-adapted geodesic spheres" in (iii) is redundant.

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